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A FLUX-VECTOR SPLITTING METHOD FOR STEADY NAVIER–STOKES EQUATIONS

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SUMMARY

The flux-vector splitting method is applied to the convective part of the steady Navier–Stokes equations for incompressible flow. By the use of partial upwind differences in the split first-order part and central differences in the second-order part, a set of discrete equations is obtained which can be solved by vector variants of classical relaxation schemes. It is shown that accurate results can be obtained on one of the GAMM backward-facing step test problems.

KEY WORDS Steady Navier–Stokes equations Flux-vector splitting Vector variants of relaxation schemes

INTRODUCTION

The flux-vector splitting method was introduced by Steger and Warming[1] to solve both unsteady and steady Euler equations. For the steady case the time-marching technique was used, i.e., integration in time of the unsteady equations up to steady state. It was shown by Jespersen[2] that the flux-vector splitting method can also be used directly on the steady Euler equations to generate discrete equations which can be solved by iterative methods. By this example it was shown that, more generally, hybrid equations, i.e. equations possessing simultaneously elliptic and hyperbolic properties, can be discretized in such a way that a solution can be obtained by classical elliptic techniques.

In this paper it is shown that the flux-vector splitting technique can be applied to the hybrid first-order part of the steady Navier–Stokes equations for incompressible flow. By taking upwind differences in the split first-order part and central differences in the second-order part of the equations, the discrete set of equations can be solved by relaxation methods.

Preliminary results using this technique have already been reported by the author[3] using full upwind differences and simple boundary conditions, i.e., prescription of function values or normal derivatives of function values. Due to the use of full upwind differences, a rather large artificial viscosity was introduced. In this paper it is shown, on one of the GAMM test cases,[4] that by the use of partial upwind differences and consistent boundary equations, i.e., equations derived from the field equations, a very accurate solution can be obtained.

UPWIND DIFFERENCING FOR SYSTEMS OF EQUATIONS

Relaxation schemes, such as Jacobi, Gauss–Seidel and successive overrelaxation schemes, are only proven for positive equations, i.e., equations of the form

$$\alpha_{ii}u_i - \sum_j \alpha_{ij}u_j = f_i, \tag{1}$$

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where, for a discretized partial differential equation, the subscript i refers to a grid point, while j refers to surrounding grid points.

The system (1) is said to be positive if it has the following properties:

1. positive coefficients: $\alpha_{ii} > 0, \alpha_{ij} \geq 0$;
2. dominance of the central node i: $\alpha_{ii} > \Sigma_{j \neq i} \alpha_{ij}$;
3. irreducibility: the system cannot be decoupled into independent subsystems.

Classical discretizations of scalar elliptic partial differential equations, as, for instance, the central discretization of the Laplace equation, generate difference equations of positive type.

However, it is clear that ellipticity of the partial differential equation is not a necessary condition to achieve a difference equation of positive type. For instance, the scalar steady advection equation

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = 0 \tag{2}$$

leads to a positive-type difference equation if upwind differencing is used, i.e., backward differencing for terms in (2) with a positive coefficient and forward differencing for terms with a negative coefficient. Hence, for $a > 0$, $b < 0$ in (2),

$$a(u_{i,j} - u_{i-1,j}) + b(u_{i,j+1} - u_{i,j}) = 0$$

or

$$[a + (-b)]u_{i,j} - au_{i-1,j} - (-b)u_{i,j+1} = 0. \tag{3}$$

Clearly equation (3) can be solved by any standard relaxation scheme.

It is rather easy to extend the notion of scalar positiveness to vector positiveness for systems of first-order equations with system matrices with real eigenvalues:

$$\mathbf{A}\frac{\partial \xi}{\partial x} + \mathbf{B}\frac{\partial \xi}{\partial y} = 0. \tag{4}$$

When **A** and **B** have real eigenvalues, it is always possible to split the matrices into a sum of a matrix with positive eigenvalues and a matrix with negative eigenvalues:

$$\mathbf{A} = \mathbf{A}^+ + \mathbf{A}^-, \quad \mathbf{B} = \mathbf{B}^+ + \mathbf{B}^-.$$

Equation (4) then can be written in split form as

$$\mathbf{A}^+\frac{\partial^+ \xi}{\partial x} + \mathbf{A}^-\frac{\partial^- \xi}{\partial x} + \mathbf{B}^+\frac{\partial^+ \xi}{\partial y} + \mathbf{B}^-\frac{\partial^- \xi}{\partial y} = 0. \tag{5}$$

An upwind discretization of (5) is then obtained when the '+' terms are discretized by backward differences and the '−' terms by forward differences:

$$\mathbf{A}^+(\xi_{i,j} - \xi_{i-1,j}) + \mathbf{A}^-(\xi_{i+1,j} - \xi_{i,j}) + \mathbf{B}^+(\xi_{i,j} - \xi_{i,j-1}) + \mathbf{B}^-(\xi_{i,j+1} - \xi_{i,j}) = 0$$

or

$$(\mathbf{A}^+ + \mathbf{B}^+ - \mathbf{A}^- - \mathbf{B}^-)\xi_{i,j} - \mathbf{A}^+\xi_{i-1,j} - (-\mathbf{A}^-)\xi_{i+1,j} - \mathbf{B}^+\xi_{i,j-1} - (-\mathbf{B}^-)\xi_{i,j+1} = 0. \tag{6}$$

Although it is not a general rule, clearly for a large class of systems the coefficient matrix $\mathbf{C} = \mathbf{A}^+ + \mathbf{B}^+ - \mathbf{A}^- - \mathbf{B}^-$ has positive eigenvalues. In this case equation (6) is a vector analogue of the scalar equation (3).

It can be said to be of vector-positive type, since the matrix coefficients then have positive eigenvalues. It is clear that vector variants of relaxation schemes can be used on vector-positive equations.

A systematic way to split the matrices **A** and **B** in (4) is obtained from the flux-vector splitting technique of Steger and Warming[1] for matrices with real eigenvalues and a complete set of eigenvectors. The splitting can then be obtained from a splitting of the eigenvalue matrices.

By denoting the eigenvalue matrices of **A** and **B** in (4) by $\mathbf{\Lambda}_A$ and $\mathbf{\Lambda}_B$ and the left eigenvector matrices by $\mathbf{X}_A$ and $\mathbf{X}_B$, obviously

$$\mathbf{A}=\mathbf{X}_A^{-1}\mathbf{\Lambda}_A\mathbf{X}_A,\quad \mathbf{B}=\mathbf{X}_B^{-1}\mathbf{\Lambda}_B\mathbf{X}_B. \tag{7}$$

The eigenvalue matrices can be split into

$$\mathbf{\Lambda}_A=\mathbf{\Lambda}_A^{+}+\mathbf{\Lambda}_A^{-},\quad \mathbf{\Lambda}_B=\mathbf{\Lambda}_B^{+}+\mathbf{\Lambda}_B^{-}, \tag{8}$$

where for **A** (similarly for **B**)

$$\mathbf{\Lambda}_A^{+}=\begin{pmatrix}\lambda_{1A}^{+} & & & \\ & \lambda_{2A}^{+} & & \\ & & \cdots & \\ & & & \lambda_{nA}^{+}\end{pmatrix},\quad \mathbf{\Lambda}_A^{-}=\begin{pmatrix}\lambda_{1A}^{-} & & & \\ & \lambda_{2A}^{-} & & \\ & & \cdots & \\ & & & \lambda_{nA}^{-}\end{pmatrix}$$

with $\lambda_{iA}^{+}=\max(\lambda_{iA},0)$ and $\lambda_{iA}^{-}=\min(\lambda_{iA},0)$. The split matrices then are obtained by

$$\mathbf{A}^{+}=\mathbf{X}_A^{-1}\mathbf{\Lambda}_A^{+}\mathbf{X}_A,\quad \mathbf{A}^{-}=\mathbf{X}_A^{-1}\mathbf{\Lambda}_A^{-}\mathbf{X}_A,\quad \mathbf{B}^{+}=\mathbf{X}_B^{-1}\mathbf{\Lambda}_B^{+}\mathbf{X}_B,\quad \mathbf{B}^{-}=\mathbf{X}_B^{-1}\mathbf{\Lambda}_B^{-}\mathbf{X}_B. \tag{9}$$

FLUX-VECTOR SPLITTING FOR STEADY NAVIER–STOKES EQUATIONS

The steady Navier–Stokes equations for an incompressible fluid are

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{\partial P}{\partial x}=\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), \tag{10}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\frac{\partial P}{\partial y}=\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right), \tag{11}$$

$$c^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0, \tag{12}$$

where u and v are the Cartesian components of velocity, c is a reference velocity introduced to homogenize the eigenvalues of the system matrix, ν is kinematic viscosity and P is pressure divided by density.

In system form the set of equations (10)–(12) becomes

$$\begin{pmatrix}u & 0 & 1\\ 0 & u & 0\\ c^2 & 0 & 0\end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix}u\\ v\\ P\end{pmatrix}+\begin{pmatrix}v & 0 & 0\\ 0 & v & 1\\ 0 & c^2 & 0\end{pmatrix}\frac{\partial}{\partial y}\begin{pmatrix}u\\ v\\ P\end{pmatrix}=\begin{pmatrix}\nu & 0 & 0\\ 0 & \nu & 0\\ 0 & 0 & 0\end{pmatrix}\Delta\begin{pmatrix}u\\ v\\ P\end{pmatrix} \tag{13}$$

or, symbolically,

$$\mathbf{A}\frac{\partial \xi}{\partial x}+\mathbf{B}\frac{\partial \xi}{\partial y}=\mathbf{D}\left(\frac{\partial^2 \xi}{\partial x^2}+\frac{\partial^2 \xi}{\partial y^2}\right). \tag{14}$$

The eigenvalues of the system matrices **A** and **B** are

$$\lambda_{1A}=u,\quad \lambda_{2A}=\frac{u+\sqrt{(u^2+4c^2)}}{2},\quad \lambda_{3A}=\frac{u-\sqrt{(u^2+4c^2)}}{2},$$

$$\lambda_{1B}=v,\quad \lambda_{2B}=\frac{v+\sqrt{(v^2+4c^2)}}{2},\quad \lambda_{3B}=\frac{v-\sqrt{(v^2+4c^2)}}{2}.$$

The left eigenvector matrices are

$$\mathbf{X}_A=\begin{pmatrix} 0 & 1 & 0\\ \dfrac{u+\sqrt{(u^2+4c^2)}}{2} & 0 & 1\\ \dfrac{u-\sqrt{(u^2+4c^2)}}{2} & 0 & 1\end{pmatrix},\quad \mathbf{X}_B=\begin{pmatrix} 1 & 0 & 0\\ 0 & \dfrac{v+\sqrt{(v^2+4c^2)}}{2} & 1\\ 0 & \dfrac{v-\sqrt{(v^2+4c^2)}}{2} & 1\end{pmatrix}.$$

Obviously, λ_{2A} and λ_{2B} are always positive, λ_{3A} and λ_{3B} are always negative, while λ_{1A} and λ_{1B} change sign with u and v.

Hence

$$\mathbf{\Lambda}_A^+=\begin{pmatrix} u^+ & & \\ & \lambda_{2A} & \\ & & 0\end{pmatrix},\quad \mathbf{\Lambda}_A^-=\begin{pmatrix} u^- & & \\ & 0 & \\ & & \lambda_{3A}\end{pmatrix},$$

$$\mathbf{\Lambda}_B^+=\begin{pmatrix} v^+ & & \\ & \lambda_{2B} & \\ & & 0\end{pmatrix},\quad \mathbf{\Lambda}_B^-=\begin{pmatrix} v^- & & \\ & 0 & \\ & & \lambda_{3B}\end{pmatrix},$$

with

$$u^+=\max(u,0),\quad u^-=\min(u,0),\quad v^+=\max(v,0),\quad v^-=\min(v,0).$$

According to the procedure of Steger and Warming, the split matrices becomes

$$\mathbf{A}^+=\mathbf{X}_A^{-1}\mathbf{\Lambda}_A^+\mathbf{X}_A=\begin{pmatrix} \alpha_1 u+a & 0 & \alpha_1\\ 0 & u^+ & 0\\ \alpha_1 c^2 & 0 & a\end{pmatrix},$$

$$\mathbf{A}^-=\mathbf{X}_A^{-1}\mathbf{\Lambda}_A^-\mathbf{X}_A=\begin{pmatrix} \alpha_2 u-a & 0 & \alpha_2\\ 0 & u^- & 0\\ \alpha_2 c^2 & 0 & -a\end{pmatrix},$$

$$\mathbf{B}^+=X_B^{-1}\mathbf{\Lambda}_B^+\mathbf{X}_B=\begin{pmatrix} v^+ & 0 & 0\\ 0 & \beta_1 v+b & \beta_1\\ 0 & \beta_1 c^2 & b\end{pmatrix},$$

$$\mathbf{B}^-=\mathbf{X}_B^{-1}\mathbf{\Lambda}_B^-\mathbf{X}_B=\begin{pmatrix} v^- & 0 & 0\\ 0 & \beta_2 v-b & \beta_2\\ 0 & \beta_2 c^2 & -b\end{pmatrix},$$

with

$$a=\frac{c^2}{\sqrt{(u^2+4c^2)}},\quad b=\frac{c^2}{\sqrt{(v^2+4c^2)}},$$

$$\alpha = \frac{u}{\sqrt{(u^2+4c^2)}}, \quad \beta = \frac{v}{\sqrt{(v^2+4c^2)}},$$

$$\alpha_1 = \frac{1+\alpha}{2}, \quad \alpha_2 = \frac{1-\alpha}{2}, \quad \beta_1 = \frac{1+\beta}{2}, \quad \beta_2 = \frac{1-\beta}{2}$$

The split form of the system (14) becomes

$$\mathbf{A}^+ \frac{\partial \xi}{\partial x} + \mathbf{A}^- \frac{\partial \xi}{\partial x} + \mathbf{B}^+ \frac{\partial \xi}{\partial y} + \mathbf{B}^- \frac{\partial \xi}{\partial y} = \mathbf{D}\left(\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2}\right). \tag{15}$$

On a rectangular grid, using upwind differences in the first-order part and central differences in the second-order part, the discretization of (15) is

$$\begin{aligned} \mathbf{C}\xi_{i,j} = {} & \mathbf{A}^+(1/\Delta x_{\mathrm{w}})\xi_{i-1,j} + (-\mathbf{A}^-)(1/\Delta x_{\mathrm{e}})\xi_{i+1,j} + \mathbf{B}^+(1/\Delta y_{\mathrm{s}})\xi_{i,j-1} \\ & + (-\mathbf{B}^-)(1/\Delta y_n)\xi_{i,j+1} + \mathbf{D}(1/\Delta x/\Delta x_{\mathrm{w}})\xi_{i-1,j} \\ & + \mathbf{D}(1/\Delta x/\Delta x_{\mathrm{e}})\xi_{i+1,j} + \mathbf{D}(1/\Delta y/\Delta y_{\mathrm{s}})\xi_{i,j-1} + \mathbf{D}(1/\Delta y/\Delta y_{\mathrm{n}})\xi_{i,j+1}, \end{aligned} \tag{16}$$

where $\mathbf{C}$ is the sum of all the matrix coefficients on the right-hand side and

$$\Delta x_{\mathrm{w}} = x_{i,j} - x_{i-1,j}, \quad \Delta x_{\mathrm{e}} = x_{i+1,j} - x_{i,j},$$

$$\Delta x_{\mathrm{s}} = x_{i,j} - x_{i,j-1}, \quad \Delta x_{\mathrm{n}} = x_{i,j+1} - x_{i,j},$$

$$\Delta x = 0{\cdot}5(\Delta x_{\mathrm{w}} + \Delta x_{\mathrm{e}}), \quad \Delta y = 0{\cdot}5(\Delta y_{\mathrm{s}} + \Delta y_{\mathrm{n}}).$$

It was verified in a numerical way that the eigenvalues of the $\mathbf{C}$ matrix are always positive. As a consequence, the set of equations (16) is a vector-positive set. It can be solved by a vector variant of any scalar relaxation scheme.

More explicitly, the set (16) can be written as

$$\begin{aligned} & E_{11}(u_{i,j} - u_{i-1,j}) + E_{12}(u_{i,j} - u_{i+1,j}) + E_{21}(u_{i,j} - u_{i,j-1}) + E_{22}(u_{i,j} - u_{i,j+1}) \\ & + E_{31}(p_{i,j} - p_{i,j-1}) + E_{32}(p_{i,j} - p_{i,j+1}) + G_{\mathrm{w}}(u_{i,j} - u_{i-1,j}) + G_{\mathrm{e}}(u_{i,j} - u_{i+1,j}) \\ & + G_{\mathrm{s}}(u_{i,j} - u_{i,j-1}) + G_{\mathrm{n}}(u_{i,j} - u_{i,j+1}) = 0, \end{aligned} \tag{17}$$

$$\begin{aligned} & F_{11}(v_{i,j} - v_{i-1,j}) + F_{12}(v_{i,j} - v_{i+1,j}) + F_{21}(v_{i,j} - v_{i,j-1}) + F_{22}(v_{i,j} - v_{i+1,j}) \\ & + F_{31}(p_{i,j} - p_{i,j-1}) + F_{32}(p_{i,j} - p_{i\,j+1}) + G_{\mathrm{w}}(v_{i,j} - v_{i-1,j}) + G_{\mathrm{e}}(v_{i,j} - v_{i+1,j}) \\ & + G_{\mathrm{s}}(v_{i,j} - v_{i,j+1}) + G_{\mathrm{n}}(v_{i,j} - v_{i,j+1}) = 0, \end{aligned} \tag{18}$$

$$\begin{aligned} & E_{31}c^2(u_{i,j} - u_{i-1,j}) + E_{32}c^2(u_{i,j} - u_{i+1,j}) + F_{31}c^2(v_{i,j} - v_{i-1,j}) + F_{32}c^2(v_{i,j} - v_{i+1,j}) \\ & + H_{\mathrm{w}}(p_{i,j} - p_{i-1,j}) + H_{\mathrm{e}}(p_{i,j} - p_{i+1,j}) + H_{\mathrm{s}}(p_{i,j} - p_{i,j-1}) + H_{\mathrm{n}}(p_{i,j} - p_{i,j+1}) = 0, \end{aligned} \tag{19}$$

where

$$\begin{aligned} E_{11} &= (a + \alpha_1 u)/\Delta x_{\mathrm{w}}, & F_{11} &= u^+/\Delta x_{\mathrm{w}}, \\ E_{12} &= (a - \alpha_2 u)/\Delta x_{\mathrm{e}}, & F_{12} &= -u^-/\Delta x_{\mathrm{e}}, \\ E_{21} &= v^+/\Delta y_{\mathrm{s}}, & F_{21} &= (b + \beta_1 v)/\Delta x_{\mathrm{w}}, \\ E_{22} &= -v^-/\Delta y_{\mathrm{n}}, & F_{22} &= (b - \beta_2 v)/\Delta x_{\mathrm{e}}, \\ E_{31} &= \alpha_1/\Delta x_{\mathrm{w}}, & F_{31} &= \beta_1/\Delta y_{\mathrm{s}}, \\ E_{32} &= -\alpha_2/\Delta x_{\mathrm{e}}, & F_{32} &= -\beta_2/\Delta y_{\mathrm{n}}, \end{aligned}$$

$$G_w = v/\Delta x/\Delta x_w, \qquad H_w = a/\Delta x_w,$$
$$G_e = v/\Delta x/\Delta x_e, \qquad H_e = a/\Delta x_e,$$
$$G_s = v/\Delta y/\Delta y_s, \qquad H_s = b/\Delta y_s,$$
$$G_n = v/\Delta y/\Delta y_n, \qquad H_n = b/\Delta y_n.$$

Since equations (17) and (18) have terms in the velocity differences from the convective part and the diffusive part in (13), a partial upwind formulation is possible for these equations. Equation (17) is then to be replaced by

$$\begin{aligned} &E_{11}[\theta_{xx}(u_{i,j}-u_{i-1,j})+0{\cdot}5(1-\theta_{xx})(u_{i+1,j}-u_{i-1,j})] \\ &+E_{12}[\theta_{xx}(u_{i,j}-u_{i+1,j})+0{\cdot}5(1-\theta_{xx})(u_{i-1,j}-u_{i+1,j})] \\ &+E_{21}[\theta_{xy}(u_{i,j}-u_{i,j-1})+0{\cdot}5(1-\theta_{xy})(u_{i,j+1}-u_{i,j-1})] \\ &+E_{22}[\theta_{xy}(u_{i,j}-u_{i,j+1})+0{\cdot}5(1-\theta_{xy})(u_{i,j-1}-u_{i,j+1})]+\ldots=0. \end{aligned} \tag{20}$$

Equation (18) is to be treated in a similar way, involving coefficients θ_{yx} and θ_{yy}. The coefficients of $u_{i-1,j}$, $u_{i+1,j}$, $u_{i,j+1}$ and $u_{i,j-1}$ in (20), when placed on the right-hand side, become

$$G_w+0{\cdot}5(E_{11}-E_{12})+0{\cdot}5\theta_{xx}(E_{11}+E_{12}),$$
$$G_e-0{\cdot}5(E_{11}-E_{12})+0{\cdot}5\theta_{xx}(E_{11}+E_{12}),$$
$$G_s+0{\cdot}5(E_{21}-E_{22})+0{\cdot}5\theta_{xy}(E_{21}+E_{22}),$$
$$G_n-0{\cdot}5(E_{21}-E_{22})+0{\cdot}5\theta_{xy}(E_{21}+E_{22}).$$

In order to guarantee the positiveness of the set of equations, these coefficients are to be positive. This leads to conditions on the upwind factors θ_{xx}, θ_{xy}, θ_{yx} and θ_{yy} involving cell Reynolds numbers. For instance, the conditions on θ_{xx} are

$$\theta_{xx} \geqslant \frac{-E_{11}+E_{12}-2G_w}{E_{11}+E_{12}}, \tag{21}$$

$$\theta_{xx} \geqslant \frac{E_{11}-E_{12}-2G_e}{E_{11}+E_{12}}. \tag{22}$$

Similar conditions apply to θ_{xy}, θ_{yx} and θ_{yy}.

It is easy to see that the conditions (21) and (22) involve cell Reynolds numbers by considering the case of a uniform grid. Then (21) and (22) become

$$\theta_{xx} \geqslant \frac{-u-2v/\Delta x}{2a+\alpha u}, \tag{23}$$

$$\theta_{xx} \geqslant \frac{u-2v/\Delta x}{2a+\alpha u}. \tag{24}$$

NUMERICAL EXAMPLES

Figure 1 shows the well known GAMM backward-facing step problem,[4] discretized with a coarse grid with 202 elements. In the actual computation a grid which was two times more refined in each direction was used. This grid has 3232 elements. The following boundary conditions were imposed.

Figure 1. Coarse computational grid for the backward-facing step problem

At inlet

$u = u_0(y)$, in which $u_0(y)$ is a parabolic profile with a mean velocity c.

$v = 0$.

p from a combination of the x-momentum equation (17) and the pressure (mass) equation (19), so that derivatives in the upstream direction are eliminated.

These equations are

$$\alpha_1 \delta_x^+ p + (\alpha_2 u - a)\delta_x^- u + \alpha_2 \delta_x^- p = \nu \frac{\partial^2 u}{\partial y^2}, \tag{25}$$

$$a\delta_x^+ p + \alpha_2 c^2 \delta_x^- u - a\delta_x^- p = 0, \tag{26}$$

where δ_x^+ and δ_x^- denote the derivatives in the backward and the forward directions. In (25) and (26) simplifications coming from an assumption of fully developed flow in the inlet section and upstream of it are already introduced: $\delta_x^+ u = 0$, $\partial p/\partial y = 0$, $\partial^2 u/\partial x^2 = 0$.

Combining (25) and (26) and eliminating $\delta_x^+ p$ gives

$$0{\cdot}5[(u - \sqrt{(u^2 + 4c^2)}]\delta_x^- u + \delta_x^- p = \nu \frac{\partial^2 u}{\partial y^2}. \tag{27}$$

At outlet

$p = 0$.

$v = 0$.

u from a combination of the x-momentum equation (17) and the pressure equation (19), so that derivatives in the downstream direction are eliminated.

Again, with a simplification from an assumption of fully developed flow in the outlet section and downstream of it, $\delta_x^- u = 0$, $\partial p/\partial y = 0$, $\partial^2 u/\partial x^2 = 0$, these equations are

$$(\alpha_1 u + a)\delta_x^+ u + \alpha_1 \delta_x^+ p + \alpha_2 \delta_x^- p = \nu \frac{\partial^2 u}{\partial y^2}, \tag{28}$$

$$\alpha_1 c^2 \delta_x^+ u + a\delta_x^+ p - a\delta_x^- p = 0. \tag{29}$$

Combining (28) and (29) gives

$$0{\cdot}5[(u + \sqrt{(u^2 + 4c^2)}]\delta_x^+ u + \delta_x^+ p = \nu \frac{\partial^2 u}{\partial y^2}. \tag{30}$$

At solid boundaries

$u = 0$.

$v = 0$.

p from a combination of the pressure equation and the momentum equations, so that derivatives in the outgoing direction are eliminated.

For instance, at the horizontal part of bottom boundary, the equations are, for $u=0$ and $v=0$,

y-momentum

$$c\,\delta_y^+ v + \delta_y^+ p - c\,\delta_y^- v + \delta_y^- p = 2\nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{31}$$

pressure

$$c\delta_x^+ u + \delta_x^+ p + c\delta_x^- u - \delta_x^- p + c\,\delta_y^+ v + \delta_y^+ p + c\,\delta_y^- v - \delta_y^- p = 0. \tag{32}$$

In equations (31) and (32) further simplifications are possible. Due to the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad \frac{\partial u}{\partial x} = 0 \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = 0, \quad \frac{\partial v}{\partial y} = 0, \quad \frac{\partial^2 v}{\partial y^2} = 0,$$

(31) and (32) become

$$c\delta_y^+ v + \delta_y^+ p + \delta_y^- p = 0, \tag{33}$$

$$\delta_x^+ p - \delta_x^- p + c\delta_y^+ v + \delta_y^+ p - \delta_y^- p = 0. \tag{34}$$

Elimination of $\delta_y^+ v$ and $\delta_y^+ p$ is reached by subtracting (33) from (34):

$$\delta_x^+ p - \delta_x^- p - 2\delta_y^- p = 0. \tag{35}$$

Similar equations can be derived at other parts of the solid boundaries.

Figure 2 shows the solution obtained for the full upwind formulation with a successive underrelaxation method (relaxation factor 0·8) in red–black ordering for

$$Re = U_{\max} h/\nu = 150, \tag{36}$$

where $U_{\max}$ is the maximum value of the velocity at the inlet section and h is the step height. The streamlines shown in Figure 2 were obtained through integration of the calculated velocity profiles. The ratio of reattachment length to step height is about 5·5. The experimental result is about 6. This shows the artificial viscosity associated with the use of full upwind differences. It was found that the scheme is unstable for the minimum values of the upwind factors calculated according to formulae like (21) and (22). In order to achieve stability, upwind factors of the following form are necessary:

$$\theta = \mu + (1-\mu)\theta_{\min}, \tag{37}$$

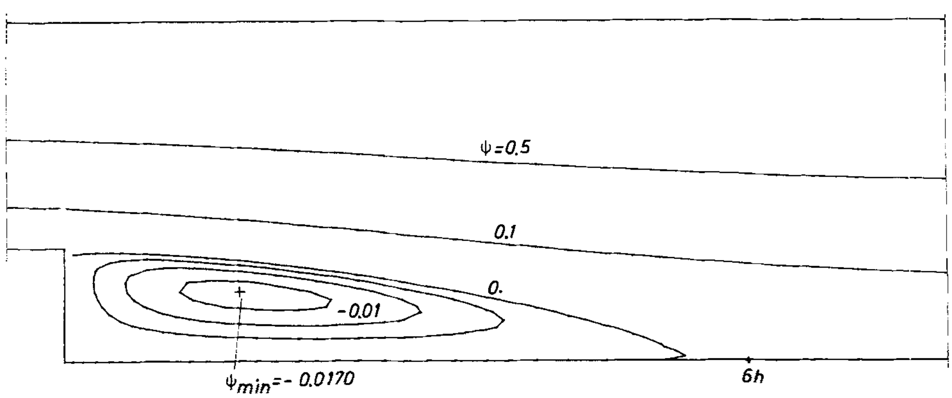

Figure 2. Streamlines for the full upwind method ($\mu = 1$) on a twice refined grid; $Re = 150$

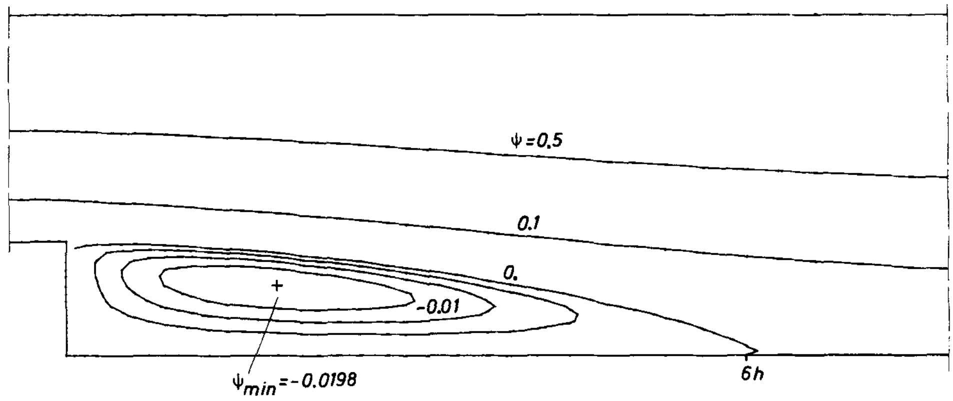

Figure 3. Streamlines for the partial upwind method ($\mu = 0{\cdot}25$) on a twice refined grid; $Re = 150$

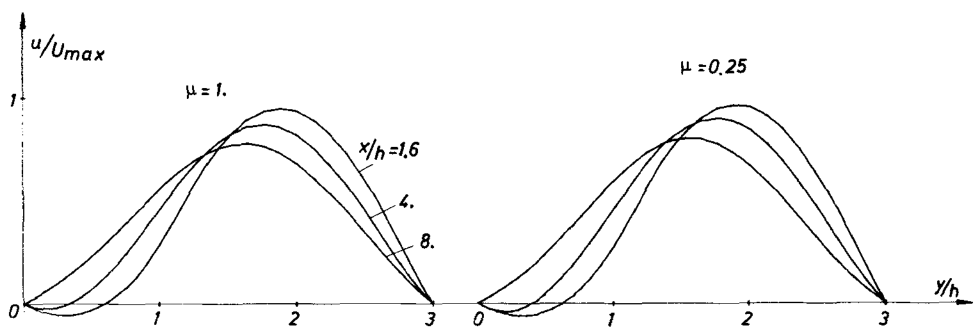

Figure 4. Longitudinal velocity profiles for the full upwind and partial upwind method

where $\theta_{\min}$ is the theoretical minimum and μ is to be at least 0·25.

Figure 3 shows the result obtained for the same conditions as in Figure 2 for $\mu = 0{\cdot}25$. The ratio of reattachment length to step height is here about 6·1, which is an almost correct result.

Figure 4 shows the longitudinal velocity profiles at the stations $1{\cdot}6h, 4h$ and $8h$ downstream of the step for the upwind factors $\mu = 1$ and $\mu = 0{\cdot}25$. The velocity profiles for $\mu = 0{\cdot}25$ almost coincide with the experimental velocity profiles.[4]

Figure 5 shows the region in which all cell Reynolds numbers are less than 2. This region covers almost the whole recirculation zone, which explains why the solution can be so accurate.

Figure 6 shows the solution obtained on a grid refined once with respect to the grid shown in Figure 1, for the same conditions as in the previous examples. The upwind factor μ had to be enlarged to 0·35 in order to obtain a solution on this grid. The figure shows streamlines obtained from the longitudinal velocity profiles, interpolated to the fine grid. The region in which the cell Reynolds numbers are less than 2 is in this example much smaller than shown in Figure 5, but still covers an essential part of the recirculation zone. This explains why the solution can have a reasonable accuracy.

On the grid shown in Figure 1, no reasonable result can be obtained for the Reynolds number 150, since for this grid cell Reynolds numbers are almost everywhere larger than 2.

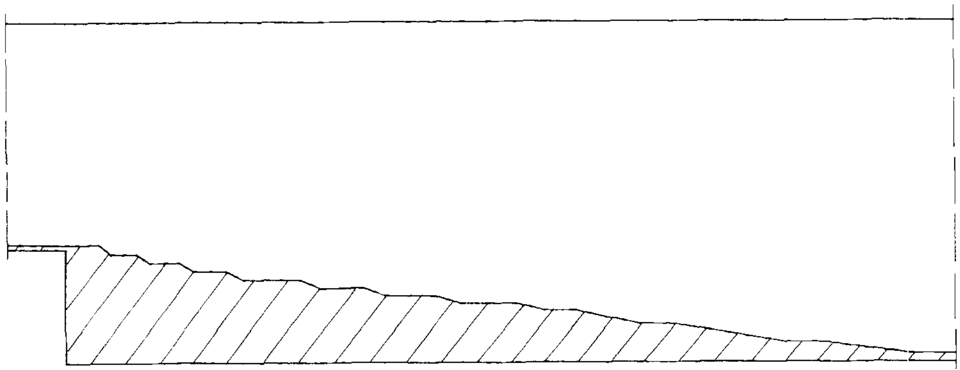

Figure 5. Region in which all cell Reynolds numbers are less than 2 for the partial upwind method on the twice refined grid; $Re = 150$

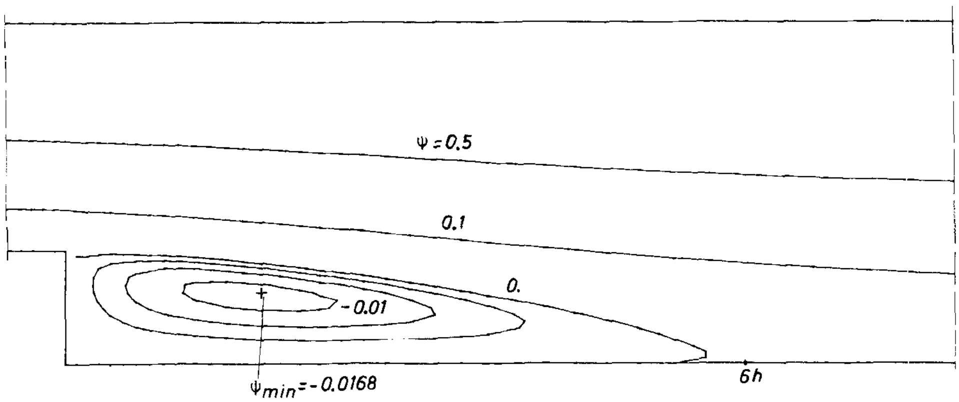

Figure 6. Streamlines for the partial upwind method ($\mu = 0{\cdot}35$) on a once refined grid; $Re = 150$

CONCLUSION

It has been shown that the flux-vector splitting technique can be applied to steady Navier–Stokes equations in incompressible flow, leading to discrete equations which can be solved by vector variants of classic relaxation schemes. By the use of partial upwind differences an accurate solution can be obtained.

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